COMBINATORIAL LEMMAS IN HIGHER DIMENSIONS

BY

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1. **Introduction.** The starting point for the investigations reported on in this paper was the following result, due to Spitzer and Widom [3], which generalized a similar result due to Kac [2].

THEOREM 1.1. *Let* $U=\{u_k;\, k=1,2,\cdots,n\}$ *be a set of n 2-dimensional vectors and let* $s_k=u_1+\cdots+u_k,\ k=1,\cdots,n$. *If* $\sigma\colon i_1,i_2,\cdots,i_n$ *is any permutation of* $1,2,\cdots,n$, *we let* $s_k(\sigma)=u_{i_1}+\cdots+u_{i_k},\ k=1,\cdots,n$. *Finally, let* $T_n(\sigma)$ *denote the length of the boundary of the convex hull* $K(\sigma)$ *of the path* $s_1(\sigma),\cdots,s_n(\sigma)$ (*i.e., the convex hull of the point-set* $0,s_1(\sigma),\ s_2(\sigma),\cdots,s_n(\sigma)$). *Then*

$$\sum_{\sigma} T_n(\sigma)=2\sum_{\sigma}\sum_{k=1}^{n}\frac{1}{k}\,|s_k(\sigma)|. \tag{1.1}$$

If we denote the sum of the vectors in a subset A of U by u_A and the number of vectors in A by n_A, then (1.1) can also be written

$$\sum_{\sigma} T_n(\sigma)=2\sum_{A}(n_A-1)!(n-n_A)!\,|u_A|. \tag{1.2}$$

Spitzer and Widom derived their result from a formula of Cauchy, expressing the length of the circumference of a compact, convex set in the plane in terms of the lengths of the projections of the set on the lines of the plane and from a lemma of Kac [2, p. 507], asserting the validity of formula (1.1) in the limiting case where the vectors in U all lie on the same line. Thus, in spite of the purely combinatorial nature of this lemma, their proof was not combinatorial. Then one of us (see [1]) succeeded in giving a combinatorial proof of Theorem 1.1 and on the basis of this result the following conjecture was made.

Let $V_n(\sigma)$ *denote the area of* $K(\sigma)$ *and let* $T(u_{A_1},u_{A_2})$ *denote the area of a triangle with sides* u_{A_1},u_{A_2} *and* $u_{A_1}+u_{A_2}$. *Then*

$$\sum_{\sigma} V_n(\sigma)=2\sum_{\sigma}{}'(n_{A_1}-1)!(n_{A_2}-1)!(n-n_{A_1}-n_{A_2})!\,T(u_{A_1},u_{A_2}) \tag{1.3}$$

where the summation sign $\sum'$ *indicates that we are summing only over non-empty, disjoint subsets* A_1 *and* A_2 *of* U.

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It turned out that the natural way to prove the conjecture was to prove the 3-dimensional analogue of Theorem 1.1, the conjecture being a limiting case thereof. This can be done by generalizing the methods and results in [1] to three dimensions. In fact, propositions of the above-mentioned kind are valid in an arbitrary number of dimensions, as shown in the sequel.

The clue to the whole investigation is the simple observation stated as Lemma 2.1 in §2. From this we derive (in §2) two combinatorial lemmas, 2.2 and 2.3, which in turn are used in §3 to prove the generalizations of the results of Spitzer-Widom and Kac and in §4 to obtain some formulas of probabilistic interest.

2. **Combinatorial lemmas.** Let $U = \{u_k;\ k = 1,\cdots,n\}$ be a set of n p-dimensional vectors and let $s_0 = 0$, $s_k = u_1 + \cdots + u_k$. If σ: $i_1,\cdots,i_n$ is any permutation of $1,2,\cdots,n$, we let $s_k(\sigma) = u_{i_1} + \cdots + u_{i_k}$. The notations u_A and n_A will represent, respectively, the sum of the vectors in a subset A of U and the number of vectors in A. The hyperplane determined by $p-1$ linearly independent p-dimensional vectors $v_1, v_2,\cdots,v_{p-1}$ will be denoted by $\gamma(v_1,\cdots,v_{p-1})$.

Definition 2.1. Let $u_1,u_2,\cdots,u_n$ be n p-dimensional vectors. We say the set $U = \{u_k;\ k = 1,\cdots,n\}$ is *skew with respect to the hyperplane* γ *of* R^p if at most one of the vectors u_A, $A \subset U$ is contained in γ.

Our fundamental lemma may now be formulated as follows.

Lemma 2.1. *Let $U = \{u_k;\ k = 1,\cdots,n\}$ be skew with respect to a $(p-1)$-dimensional hyperplane γ containing $s_n = u_U$. Let H denote one of the half-spaces detrmined by γ. Then, there exists exactly one cyclic permutation σ of $1,2,\cdots,n$ such that exactly m $(m = 0,1,\cdots,n-1)$ of the points $s_1(\sigma),\cdots,s_{n-1}(\sigma)$ lie in H. (See Figure 2.1 for an example with $m = n-1$.)*

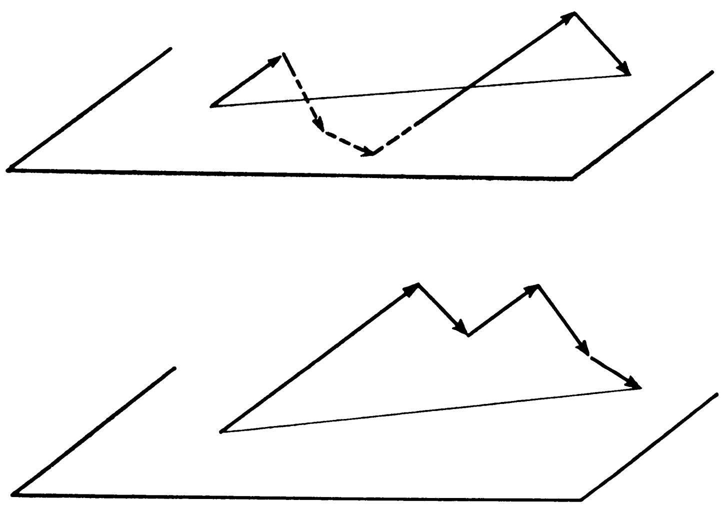

Figure 2.1

Proof. Let e be the unit vector in H perpendicular to γ. Since U is skew, there exists one and only one index k such that exactly m of the inner products (e,s_0), $(e,s_1),\cdots,(e,s_{n-1})$ are greater than (e,s_k). The permutation σ: $k+1,\cdots,n,\ 1,\cdots,k$ has the property stated in the lemma. The uniqueness of σ follows from the uniqueness of k.

We need another definition.

DEFINITION 2.2. Let $U=\{u_k;\ k=1,2,\cdots,n\}$ be a set of n p-dimensional vectors u_k. We say the set is *skew* if every p-tuple of vectors $u_{A_1},u_{A_2},\cdots,u_{A_p}$ corresponding to (nonempty) disjoint subsets $A_1,\cdots,A_p$ of U form a set of linearly independent vectors.

Let $U=\{u_k;\ k=1,\cdots,n\}$ be a fixed, skew set of p-dimensional vectors and let σ be any permutation of $1,2,\cdots,n$. Clearly, any edge in the convex hull of the path

$$s_1(\sigma),s_2(\sigma),\cdots,s_n(\sigma) \tag{2.1}$$

is the sum of the vectors in some subset A of U, and this subset is uniquely determined by the edge since U is skew. Now, let $A_1,A_2,\cdots,A_r$ be r disjoint, nonempty subsets of U, $r=p-2$ or $p-1$. Lemma 2.1 enables us to count exactly how many times the whole path

$$u_{A_1},u_{A_1}+u_{A_2},\cdots,u_{A_1}+\cdots+u_{A} \tag{2.2}$$

lies in the boundary of the convex hull of (2.1), as σ ranges over all permutations of $1,2,\cdots,n$, i.e., how many times $u_{A_1},u_{A_2},\cdots,u_{A_r}$ are simultaneously edges such that the endpoint of u_{A_ρ} coincides with the startpoint of $u_{A_{\rho+1}}$, $\rho=1,2,\cdots,r-1$. To avoid having to adopt a convention for degenerate cases we assume from now on that $n\geqq p$.

LEMMA 2.2. *Let $U=\{u_k;\ k=1,\cdots,n\}$ be a fixed, skew set of n p-dimensional vectors $(p>1)$ and let $A_1,\cdots,A_{p-1}$ be fixed, nonempty and disjoint subsets of U. Then the whole path*

$$u_{A_1},u_{A_1}+u_{A_2},\cdots,u_{A_1}+u_{A_2}+\cdots+u_{A_{p-1}} \tag{2.3}$$

appears in the boundary of the hull of exactly

$$2(n_{A_1}-1)!(n_{A_2}-1)!\cdots(n_{A_{p-1}}-1)!(n-n_{A_1}-\cdots-n_{A_{p-1}})! \tag{2.4}$$

of the $n!$ paths $s_1(\sigma)$, $s_2(\sigma),\cdots,s_n(\sigma)$ as σ ranges over all permutations.

Proof. Let $u_0=-s_n$ and let A_0 denote the complement of $A_1\cup A_2\cup\cdots\cup A_{p-1}$ in $U_0=\{u_1,u_2,\cdots,u_n,u_0\}$. We call

$$s_1(\sigma),s_2(\sigma),\cdots,s_n(\sigma),u_0 \tag{2.5}$$

the completed path associated with $u_{i_1}, u_{i_2}, \cdots, u_{i_n}$. Let H_1 and H_2 be the two halfspaces in R determined by the hyperplane $\gamma = \gamma(u_{A_1}, u_{A_2}, \cdots, u_{A_{p-1}})$. We include γ in both H_1 and H_2. We can think of any completed path (2.5) whose hull-boundary contains (2.3) as subdivided naturally into p ordered sets of vectors $A_1(\sigma), A_2(\sigma), \cdots, A_{p-1}(\sigma)$ and $A_0(\sigma)$, $A_i(\sigma)$ being the set A_i with the ordering induced by σ $(i = 0, 1, \cdots, p-1)$. The paths corresponding to each of these ordered sets of vectors must lie either all in H_1 or all in H_2. Moreover, any ordering of the vectors in $A_1, \cdots, A_{p-1}$ and A_0 subject to the condition that their paths lie in the same half-space H_j $(j = 1, 2)$ gives rise to a completed path $s_1(\sigma), \cdots, s_n(\sigma), u_0$ the origin being determined from the position of u_0 in the ordering of A_0. Thus we need only to count how many different orderings of vectors in $A_1, \cdots, A_{p-1}$ and A_0 there are such that all p subpaths lie in the same half-space H_j.

The sets $A_1, \cdots, A_{p-1}$ and A_0 are all skew with respect to $\gamma(u_{A_1}, \cdots, u_{A_{n-1}})$ since U is skew. Hence, from Lemma 2.1, we find that there are

$$(n_{A_1} - 1)! \cdots (n_{A_{n-1}} - 1)!(n - n_{A_1} - \cdots - n_{A_{p-1}})!$$

different ways of ordering $A_1, \cdots, A_{p-1}$ and A_0 such that all p subpaths lie in H_1, say. Taking into account also H_2, the proof is completed.

Let $B_1, B_2, \cdots, B_{p-1}$ be $p-1$ subsets of U. We say the ordered $(p-1)$-tuple $B_1, \cdots, B_{p-1}$ has the property (*) (with respect to $A_1, \cdots, A_{p-2}$) if

(*) $B_1, \cdots, B_{p-1}$ are nonempty and disjoint and there exist indices $\nu_1, \nu_2, \cdots, \nu_{p-2}$ such that $0 \leqq \nu_1 < \nu_2 < \cdots < \nu_{p-2} \leqq p-1$ and $B_{\nu_1+1} \cup B_{\nu_2} = A_1$, $B_{\nu_2+1} \cup B_{\nu_3} = A_2, \cdots, B_{\nu_{n-3}+1} \cup B_{\nu_{n-2}} = A_{p-2}$.

In other words each A_k is one of the B_k's, with the possible exception that exactly one of the A_k's is the union of two B_k's.

LEMMA 2.3. *Let $U = \{u_k;\ k = 1, 2, \cdots, n\}$ be a fixed, skew set of p-dimensional vectors and let $A_1, A_2, \cdots, A_{p-2}$ be $p-2$ fixed, nonempty and disjoint subsets of U. Then, the whole path*

$$u_{A_1}, u_{A_1} + u_{A_2}, \cdots, u_{A_1} + \cdots + u_{A_{p-2}} \tag{2.6}$$

lies in the boundary of the hull of exactly

$$\sum{}^*(n_{B_1} - 1)!(n_{B_2} - 1)! \cdots (n_{B_{n-1}} - 1)!(n - n_{B_1} - \cdots - n_{B_{p-1}})! \tag{2.7}$$

of the $n!$ paths $s_1(\sigma), \cdots, s_n(\sigma)$ as σ ranges over all permutations. Here $\sum^*$ indicates that we are summing only over ordered $(p-1)$-tuples $B_1, \cdots, B_{p-1}$ with the property (*).

Proof. Let $B_1, B_2, \cdots, B_{p-1}$ be $p-1$ subsets of U satisfying (*). If the path $u_{B_1}, u_{B_1} + u_{B_2}, \cdots, u_{B_1} + \cdots + u_{B_{p-1}}$ appears in the boundary of the hull of $s_1(\sigma), \cdots, s_n(\sigma)$, then so does (2.6). Conversely, assume that the path (2.6) appears in the boundary of the convex hull of $s_1(\sigma), \cdots, s_n(\sigma)$ and let us consider one of the two sides (of dimension $p-1$) of the hull which contains (2.6). Since U is skew, the side contains exactly p vertices of the path $s_1(\sigma), \cdots, s_n(\sigma)$. Let these vertices be numbered $1, 2, \cdots, p$ in accordance with their ordering on the path and let B_i, $i = 1, 2, \cdots, p-1$ denote the subset of U corresponding to the vector from vertex number i to vertex number $i+1$. Then $B_1, \cdots, B_{p-1}$ has the property (*). We may therefore determine the number $n(A_1, \cdots, A_{p-2})$ of times the path $u_{A_1}, u_{A_1} + u_{A_2}, \cdots, u_{A_1} + \cdots + u_{A_{p-2}}$ appears in the boundary of the hull of $s_1(\sigma), \cdots, s_n(\sigma)$ as follows. To any fixed, ordered $(p-1)$-tuple $B_1, \cdots, B_{p-1}$ with the property (*) we count how many times the path $u_{B_1}, \cdots, u_{B_1} + \cdots + u_{B_{p-1}}$ appears in the boundary of the hull of $s_1(\sigma), \cdots, s_n(\sigma)$ as σ ranges over all permutations. Let us denote this number by $n(B_1, \cdots, B_{p-1})$. Then

$$n(A_1, \cdots, A_{p-2}) = \frac{1}{2} \sum{}^{*} n(B_1, \cdots, B_{p-1}) \tag{2.8}$$

where $\sum^*$ indicates that we are summing only over ordered $(p-1)$-tuples $B_1, \cdots, B_{p-1}$ satisfying (*).

According to Lemma 2.2 we have

$$n(B_1, \cdots, B_{p-1}) = 2(n_{B_1} - 1)! \cdots (n_{B_{p-1}} - 1)!(n - n_{B_1} - \cdots - n_{B_{p-1}})!. \tag{2.9}$$

3. **Volume and surface area of convex polyhedrons.** In this section we derive from Lemma 2.2 a number of combinatorial formulas relating to the volume and surface area of p-dimensional convex polyhedrons. Let again $U = \{u_k;\ k = 1, \cdots, n\}$ be a fixed skew set of n p-dimensional vectors and let σ be any permutation of $1, 2, \cdots, n$. We shall denote the volume and the surface area of the convex hull $K(\sigma)$ of the path $s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma)$ by $V_n(\sigma)$ and $T_n(\sigma)$, respectively. The $(p-1)$-dimensional measure of the convex hull of the path $u_{A_1}, u_{A_1} + u_{A_2}, \cdots, u_{A_1} + \cdots + u_{A_{p-1}}$ where $A_p, \cdots, A_{p-1}$ is any set of $p-1$ non-empty and disjoint subsets of U, will be denoted by $T(u_{A_1}, u_{A_2}, \cdots, u_{A_{p-1}})$.

In order to clarify the exposition we have chosen to discuss the cases $p = 1, 2$ and 3 before passing over to the general results.

Let $p = 2$. Then $T_n(\sigma)$ is the length of the boundary of $K(\sigma)$ and we have the following formula due to Spitzer and Widom:

$$\sum_{\sigma} T_n(\sigma) = 2 \sum_{A} (n_A - 1)!(n - n_A)! \, |u_A|, \tag{3.1}$$

where $|u_A|$ is the length of the vector u_A.

As was pointed out in [1], this relation may be proved as follows. By definition, $T_n(\sigma)$ equals the sum of the lengths of the edges in $K(\sigma)$. It was mentioned in §2 that each edge in $K(\sigma)$ is the sum of the vectors in a uniquely determined subset A of U. According to Lemma 2.2, the vector u_A appears as an edge in exactly

$2(n_A - 1)!(n - n_A)!$ of the convex hulls $K(\sigma)$ as σ ranges over all permutations. This proves the formula (3.1).

Next, let $p = 3$. In this case, $T_n(\sigma)$ equals the surface area of the 3-dimensional polyhedron $K(\sigma)$, $T(u_{A_1}, u_{A_2})$ is the area of a triangle with sides u_{A_1}, u_{A_2}, and $u_{A_1} + u_{A_2}$, and

$$(3.2) \qquad \sum_{\sigma} T_n(\sigma) = 2 \sum{}' (n_{A_1} - 1)!(n_{A_2} - 1)!(n - n_{A_1} - n_{A_2})!\, T(u_{A_1}, u_{A_2})$$

where the summation sign $\sum'$ indicates that we are summing only over ordered pairs of nonempty, disjoint subsets A_1 and A_2 of U. The proof of this relation is completely analogous to that of (3.1). Formula (3.2) can be reduced to a form more in line with the formulas of Kac [2] and Spitzer-Widom [3]:

$$(3.2') \qquad \sum_{\sigma} T_n(\sigma) = 2 \sum_{\sigma} \sum_{k,m=1}^{n} \frac{1}{km} \hat{T}(s_k(\sigma), s_m(\sigma)),$$

where $\hat{T}(s_k(\sigma), s_m(\sigma))$ is the area of the triangle with vertices at $0, s_k(\sigma), s_m(\sigma)$. The general theorem is as follows.

THEOREM 3.1. *Let $U = \{u_k;\ k = 1, \cdots, n\}$ be a fixed, skew set of p-dimensional vectors, let $T_n(\sigma)$ be the surface area of the convex hull of the path $s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma)$ and let $T(u_{A_1}, \cdots, u_{A_{n-1}})$ be the $(p-1)$-dimensional measure of the convex hull of the path $u_{A_1}, u_{A_1} + u_{A_2}, \cdots, u_{A_1} + \cdots + u_{A_{-1}}$. Then*

$$(3.3) \qquad \sum_{\sigma} T_n(\sigma) = 2 \sum{}' (n_{A_1} - 1)! \cdots (n_{A_{p-1}} - 1)!(n - n_{A_1} - \cdots - n_{A_{p-1}})!\, T(u_{A_1}, \cdots, u_{A_{n-1}})$$

where the summation sign $\sum'$ indicates that we are summing only over ordered $(p-1)$-tuples of nonempty, disjoint subsets $A_1, A_2, \cdots, A_{p-1}$ of U.

Proof. The left side in (3.3) equals the sum of the areas with $(p-1)$-dimensional measures of the sides in the convex hulls $K(\sigma)$, where σ ranges over all permutations. Each side in a given $K(\sigma)$ is the convex hull of one and only one subpath $u_{A_1}, u_{A_1} + u_{A_2}, \cdots, u_{A_1} + \cdots + u_{A_{p-1}}$ of $s_1(\sigma), s_2(\sigma), \cdots, s_n(\sigma)$, where $A_1, A_2, \cdots, A_{p-1}$ are nonempty and disjoint. Lemma 2.2 tells us that this particular subpath will appear in the boundary of exactly

$$2(n_{A_1} - 1)! \cdots (n_{A_{p-1}} - 1)!(n - n_{A_1} - \cdots - n_{A_{n-1}})!$$

of the hulls $K(\sigma)$ as σ ranges over all permutations. This completes the proof.

Let $p = 1$. In their proof of formula (3.1), Spitzer and Widom exploited a result of M. Kac [2] to the effect that

$$(3.4) \qquad \sum_{\sigma} \left(\max_{0 \le k \le n} s_k(\sigma) - \min_{0 \le k \le n} s_k(\sigma) \right) = \sum_{\sigma} \sum_{k=1}^{n} \frac{1}{k} |s_k(\sigma)|.$$

Formula (3.4) is in fact a limiting case of (3.1). To see this, we note that the right hand side of (3.4) may be written as

$$\sum_{\sigma}\left(\max_{0\leqq k\leqq n} s_k(\sigma) - \min_{0\leqq k\leqq n} s_k(\sigma)\right) = \sum_{A}(n_A-1)!(n-n_A)!\,|u_A|. \tag{3.5}$$

To any set $U=\{u_k;\ k=1,\cdots,n\}$ of (nondegenerate) 1-dimensional vectors we may find a skew set $Z=\{z_k;\ k=1,\cdots,n\}$ of 2-dimensional vectors z_k such that the projection of z_k on the line containing the vectors in U is equal to u_k, $k=1,\cdots,n$. We may actually find a whole sequence $Z^{(1)},Z^{(2)},\cdots,Z^{(i)},\cdots$ of sets $Z^{(i)}=\{z_k^{(i)};\ k=1,\cdots,n\}$ of the above-mentioned type such that $z_k^{(i)}\to u_k$ as $i\to\infty$ $(k=1,\cdots,n)$. For each set $Z^{(i)}$, a formula like (3.1) holds; in symbols

$$\sum_{\sigma} T_n^{(i)}(\sigma) = 2\sum_{A}(n_A-1)!(n-n_A)!\,T(z_A^{(i)}). \tag{3.6}$$

As i approaches infinity we have

$$T_n^{(i)}(\sigma)\to 2\left(\max_{0\leqq k\leqq n} s_k(\sigma) - \min_{0\leqq k\leqq n} s_k(\sigma)\right) \tag{3.7}$$

and

$$T(z_A^{(i)})\to T(u_A)=|u_A| \tag{3.8}$$

and this verifies (3.5) (and hence (3.4)).

Now, let $p=2$. We have the following 2-dimensional analogue of formula (3.5):

$$\sum_{\sigma} V_n(\sigma) = \sum{}'(n_{A_1}-1)!(n_{A_2}-1)!(n-n_{A_1}-n_{A_2})!\,T(u_{A_1},u_{A_2}) \tag{3.9}$$

where, by definition, $V_n(\sigma)$ is the area of the convex hull of $s_1(\sigma),s_2(\sigma),\cdots,s_n(\sigma)$. (The summation sign $\sum{}'$ indicates the sum over all ordered and nonempty, disjoint subsets A_1 and A_2 of U.) In fact, somewhat more is true. Suppose that to every given ordered pair A_1,A_2 of nonempty, disjoint subsets of U, we have $(n_{A_1}-1)!(n_{A_2}-1)!(n-n_{A_1}-n_{A_2})!$ triangles with sides u_{A_1} and u_{A_2}. It is then possible to arrange all these triangles into $n!$ groups, each group corresponding to one of the $n!$ permutations σ, in such a way that within each group the triangles may be fitted together so as to cover exactly the convex hull of the path determined by the permutation which characterizes the group.

If, for instance, the set U consists of the four vectors below, and if we have

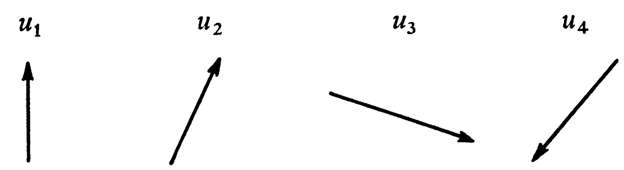

FIGURE 3.1

four triangles with sides u_1 and u_2 (two corresponding to the pair u_1, u_2 and two to the pair u_2, u_1), four with sides u_1 and u_3, etc., two triangles with sides u_1 and $u_2 + u_3$, etc., then these may be fitted together in a (by no means unique) way, so as to cover exactly and at the same time all the $n! = 24$ convex hulls generated by $u_1, \cdots, u_4$. An example of such a covering is shown in Figure 3.2, where the triangles have been numbered with congruent triangles having the same number. Table 3.1 reveals which vectors determine a given triangle.

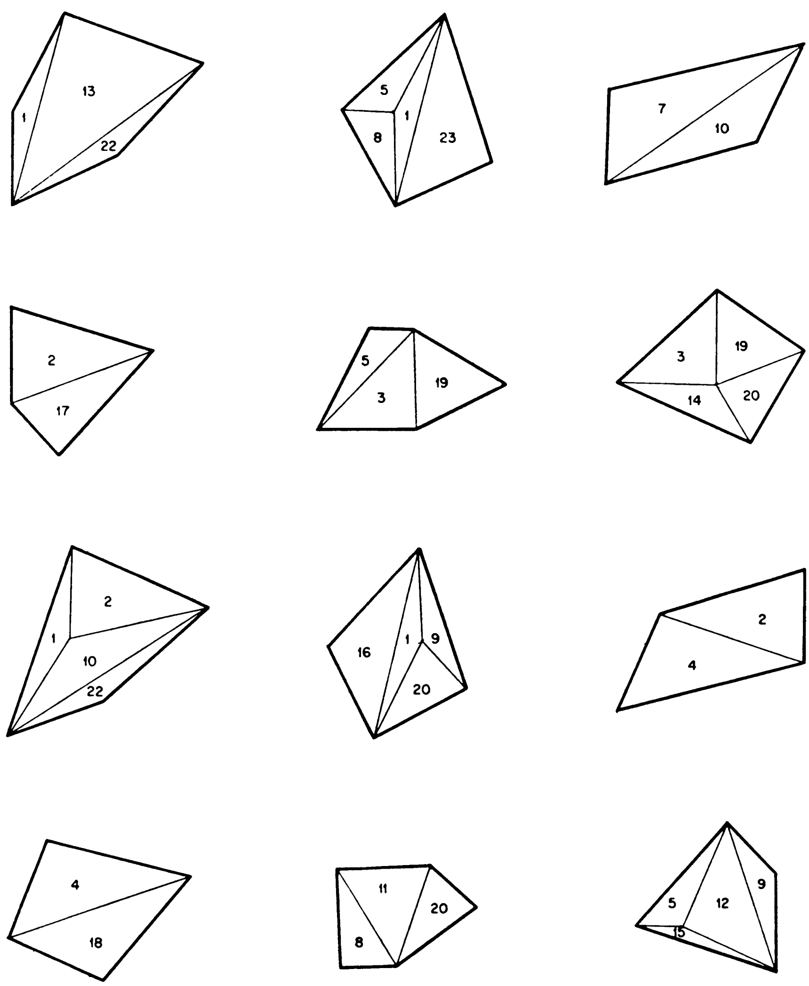

FIGURE 3.2a

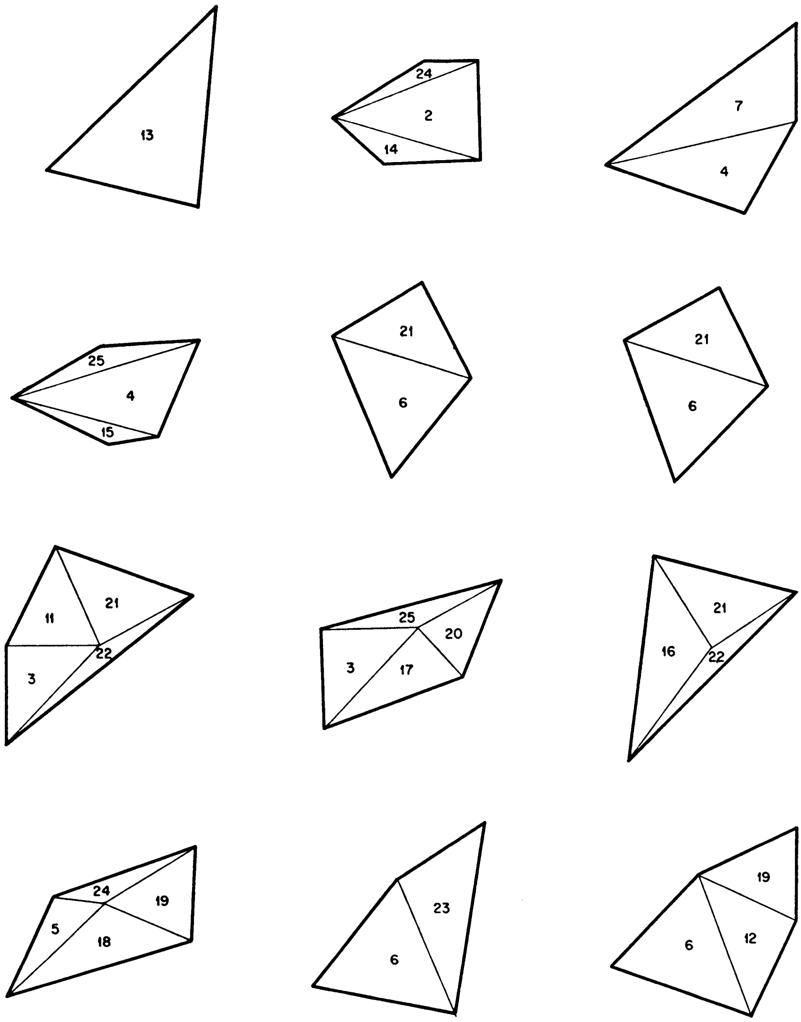

FIGURE 3.2b

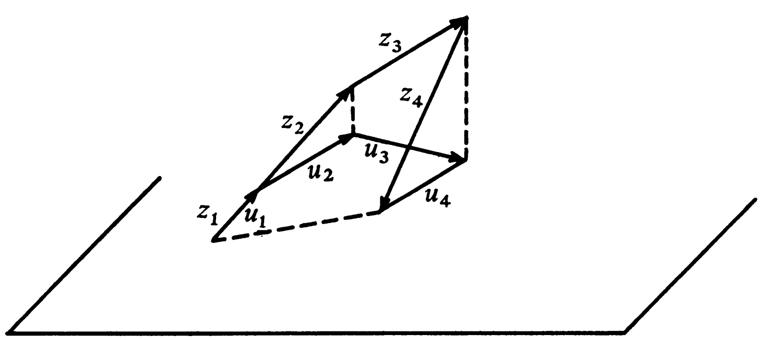

FIGURE 3.3

TABLE 3.1

triangle no.	determined by	triangle no.	determined by
1	u_1, u_2	14	$u_3, u_1 + u_4$
2	u_1, u_3	15	$u_3, u_2 + u_4$
3	u_1, u_4	16	$u_4, u_1 + u_2$
4	u_2, u_3	17	$u_4, u_1 + u_3$
5	u_2, u_4	18	$u_4, u_2 + u_3$
6	u_3, u_4	19	$u_1, u_2 + u_3 + u_4$
7	$u_1, u_2 + u_3$	20	$u_2, u_1 + u_3 + u_4$
8	$u_1, u_2 + u_4$	21	$u_3, u_1 + u_2 + u_4$
9	$u_1, u_3 + u_4$	22	$u_4, u_1 + u_2 + u_3$
10	$u_2, u_1 + u_3$	23	$u_1 + u_2, u_3 + u_4$
11	$u_2, u_1 + u_4$	24	$u_1 + u_3, u_2 + u_4$
12	$u_2, u_3 + u_4$	25	$u_1 + u_4, u_2 + u_3$
13	$u_3, u_1 + u_2$		

To see this, let us consider a skew set Z of 4 3-dimensional vectors z_1, z_2, z_3 and z_4 such that z_k projects into u_k, $k = 1,\cdots,4$. In our example the z-vectors have the coordinates $z_1 = (0,2,0)$, $z_2 = (1,2,1)$, $z_3 = (3,-1,2)$ and $z_4 = (-2,-2,-3)$ (see Figure 3.3). Let e be a unit vector perpendicular to the plane γ spanned by $u_1,\cdots,u_4$ and let $K_0(\sigma)$, σ: i_1, i_2, i_3, i_4, be the convex hull of the path z_{i_1}, $z_{i_1} + z_{i_2}$, $z_{i_1} + z_{i_2} + z_{i_3}$, $z_{i_1} + z_{i_2} + z_{i_3} + z_{i_4}$. We provide each of the triangular sides of $K_0(\sigma)$ with a unit vector f perpendicular to the side and pointing away from $K_0(\sigma)$. Let us call a side with unit vector f red if the inner product (e,f) is positive and white if (e,f) is negative ((e,f) cannot be a zero since Z is skew). The red triangles project onto γ into (nonoverlapping) triangles fitting together so as to cover exactly the convex hull of $u_{i_1},\cdots,u_{i_1}+\cdots+u_{i_4}$. Each of the projected triangles is the convex hull of one and only one subpath u_{A_1}, $u_{A_1} + u_{A_2}$ of $u_{i_1},\cdots,$ $u_{i_1} + \cdots + u_{i_4}$ and is the projection of the convex hull of the uniquely determined subpath of $z_{i_1},\cdots,z_{i_1} + \cdots + z_{i_4}$ which projects into u_{A_1}, $u_{A_1} + u_{A_2}$. According to Lemma 2.2 this subpath of $z_{i_1},\cdots,z_{i_1} + \cdots + z_{i_4}$ lies in the boundary of exactly $2(n_{A_1} - 1)!(n_{A_2} - 1)!(n - n_{A_1} - n_{A_2})!$ of the convex hulls $K_0(\sigma)$, there determining either a red or a white side of $K_0(\sigma)$. As may be seen from the proof of that lemma, the subpath must give rise to equally many red and white sides. This proves the assertion.

Note that the decomposition of the hulls into triangles as described above is *not* unique. There are several possible decompositions depending on the values of the third components of the vectors $z_1,\cdots,z_4$. The appearance of a given triangle in one of the hulls indicates that a certain set of inequalities is satisfied. If the third components are altered, the inequalities may be reversed, in which case a new decomposition will arise. Nonetheless, the total set of triangles remains unaltered.

We now formulate the general result, which may be proved by a straightforward modification of the argument given above for the case $p=2$ and $n=4$.

THEOREM 3.2. *Let $U=\{u_k;\, k=1,\cdots,n\}$ be a fixed, skew set of p-dimensional vectors. Suppose that to every given ordered set $A_1,A_2,\cdots,A_p$ of p nonempty, disjoint subsets of U, we have*

$$(n_{A_1}-1)!(n_{A_2}-1)!\cdots(n_{A_n}-1)!(n-n_{A_1}-\cdots-n_{A_n})!$$

copies of the convex hull of the path $u_{A_1}, u_{A_1}+u_{A_2},\cdots,u_{A_1}+\cdots+u_{A_n}$. It is then possible to arrange all these convex polyhedrons into n! groups in at least one way such that there is a one-to-one correspondence between the groups and the n! permutations and such that within each group the polyhedrons may be fitted together so as to yield the convex hull of the path $s_1(\sigma),s_2(\sigma),\cdots,s_n(\sigma)$ determined by the permutation which characterizes the group.

Denoting the volume of the convex hull of the path $s_1(\sigma),\cdots,s_n(\sigma)$ by $V_n(\sigma)$ and the volume of the convex hull of the path $u_{A_1},\cdots,u_{A_1}+\cdots+u_{A_n}$ by $T(u_{A_1},\cdots,u_{A_n})$ we consequently have

$$\sum_\sigma V_n(\sigma)=\textstyle\sum'(n_{A_1}-1)!\cdots(n_{A_n}-1)!(n-n_{A_1}-\cdots-n_{A_n})!\; T(u_{A_1},\cdots,u_{A_p}) \tag{3.10}$$

where the summation sign $\sum'$ indicates that we are summing only over nonempty, disjoint subsets $A_1,A_2,\cdots,A_p$ of U. Theorem 3.2 remains true even if $U=\{u_k;\, k=1,\cdots,n\}$ is not skew.

4. **Applications to probability theory.** The formulas given in §§2 and 3 may be used to obtain results of probabilistic interest. We illustrate this by some examples.

Let $X_k=(X_k^{(1)},\cdots,X_k^{(p)});\ k=1,\cdots,n$, be a set of n independent and identically distributed p-dimensional random variables. We assume for convenience that $X_k^{(1)},\cdots,X_k^{(p)}$ have a joint density function; this implies that with probability one $X_1,\cdots,X_n$ is skew. If σ: $i_1,\cdots,i_n$ is any permutation of $1,\cdots,n$, we let $S_0(\sigma)=0$, $S_1(\sigma)=X_{i_1},\cdots,S_n(\sigma)=X_{i_1}+\cdots+X_{i_n}$ and we denote the convex hull of the path $S_1(\sigma),\cdots,S_n(\sigma)$ by $K(\sigma)$. We shall indicate the identity permutation σ: $1,2,\cdots,n$ by deleting the symbol (σ) from our notations. The expectation of a random variable X will be denoted by $E\{X\}$.

EXAMPLE 4.1. Let $H_n(\sigma)$ be the number of sides in $K(\sigma)$. Then, by the identical distribution property we have $E\{H_n\}=E\{H_n(\sigma)\}$ for any permutation σ and hence

$$E\{H_n\}=\frac{1}{n!}E\left\{\sum_\sigma H_n(\sigma)\right\}. \tag{4.1}$$

The summation on the right in (4.1) equals the total number of sides in the $n!$ hulls $K(\sigma)$ as σ ranges over all permutations.

Thus, if $p = 2$, we have according to Lemma 2.2 (with probability one)

$$\sum_{\sigma} H_n(\sigma) = 2\sum_{A} (n_A - 1)!(n - n_A)! \tag{4.2}$$

$$= 2\sum_{k=1}^{n} \binom{n}{k}(k-1)!(n-k)!$$

$$= 2n!\sum_{k=1}^{n} \frac{1}{k}.$$

Hence

$$E\{H_n\} = 2\sum_{k=1}^{n} \frac{1}{k} \sim 2 \log n. \tag{4.3}$$

Similarly, if $p = 3$, we find from Lemma 2.2

$$E\{H_n\} = 2\sum_{k=1}^{n-1} \sum_{m=1}^{n-k} \frac{1}{k \cdot m} \sim 2(\log n)^2. \tag{4.4}$$

EXAMPLE 4.2. Let $p = 3$ and let $I_n(\sigma)$ denote the number of edges in $K(\sigma)$. Then

$$E\{I_n\} = 3\sum_{k=1}^{n-1} \sum_{m=1}^{n-k} \frac{1}{k \cdot m} \sim 3(\log n)^2. \tag{4.5}$$

This result may be derived from Lemma 2.3 by a reasoning similar to that given in Example 4.1. It follows, however, immediately from formula (4.4) and the fact that, if $X_1, \cdots, X_n$ is skew, then for every σ the sides in $K(\sigma)$ are triangles and hence we must have $I_n(\sigma) = (3/2)H_n(\sigma)$ and consequently $E\{I_n\} = (3/2)E\{H_n\}$.

EXAMPLE 4.3. Let $p = 2$. From Theorem 3.1 we find the following formula (due to Spitzer and Widom [3]) for the expectation of the length T_n of the boundary of K

$$E\{T_n\} = \frac{1}{n!}E\left\{\sum_{\sigma} T_n(\sigma)\right\} = \frac{2}{n!}\sum_{A} (n_A - 1)!(n - n_A)!\,E\{T(X_A)\}$$

$$= \frac{2}{n!}\sum_{k=1}^{n} \binom{n}{k} (k-1)!(n-k)!\,E\{T(S_k)\} \tag{4.6}$$

$$= 2\sum_{k=1}^{n} \frac{1}{k} E|S_k|.$$

In the case $p = 3$ we have

$$
\begin{aligned}
E(T_n) &= \frac{2}{n!}\Sigma'(n_{A_1}-1)!(n_{A_2}-1)!(n-n_{A_1}-n_{A_2})!\,E\{T(X_{A_1},X_{A_2})\} \\
&= \frac{2}{n!}\sum_{k=1}^{n-1}\sum_{m=1}^{n-k}\frac{n!}{k!\,m!\,(n-k-m)!}(k-1)!(m-1)!(n-k-m)! \\
&\qquad\qquad \cdot E\{T(S_k,S_{k+m}-S_k)\} \\
&= 2\sum_{k=2}^{n}\sum_{m=1}^{k-1}\frac{1}{m(k-m)}E\{T(S_m,S_k-S_m)\}.
\end{aligned}
\tag{4.7}
$$

EXAMPLE 4.4. As a last example we consider for $p=2$ the area V_n of the convex hull K. From formula (3.10) we obtain in the same manner as above

$$
E(V_n)=\sum_{k=2}^{n}\sum_{m=1}^{k-1}\frac{1}{m(k-m)}E\{T(S_m,S_k-S_m)\}.
\tag{4.8}
$$

If, in particular, the components $X_k^{(1)}$ and $X_k^{(2)}$ of X_k have expectation 0 and finite second moment, then $n^{-1/2}S_n$ has a bivariate normal limiting distribution and $|n^{-1/2}S_n|$ is uniformly integrable in n. Thus

$$
\begin{aligned}
&m^{-1/2}(k-m)^{-1/2}E\{T(S_m,S_k-S_m)\} \\
&\qquad = E\{T(m^{-1/2}S_m,(k-m)^{-1/2}(S_k-S_m)\} \\
&\qquad \to c \quad \text{as} \quad \begin{cases} m\to\infty, \\ k-m\to\infty, \end{cases}
\end{aligned}
\tag{4.9}
$$

where c is a finite constant. In the case of rotational symmetry of the limiting distribution it is easy to see that $2c=E\{X_k^{(1)^2}\}=E\{X_k^{(2)^2}\}$. It follows, by elementary calculations, that

$$
\frac{E\{V_n\}}{n}\to c\pi \qquad \text{as} \qquad n\to\infty.
\tag{4.10}
$$

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